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COLORING GRAPHS WITH NO INDUCED SUBDIVISION OF K_4^+

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Abstract. Let K_4^+ be the 5-vertex graph obtained from K_4, the complete graph on four vertices, by subdividing one edge precisely once (i.e. by replacing one edge by a path on three vertices). We prove that if the chromatic number of some graph G is much larger than its clique number, then G contains a subdivision of K_4^+ as an induced subgraph.

Given a graph H, a *subdivision* of H is a graph obtained from H by replacing some edges of H (possibly none) by paths. We say that a graph G contains an *induced subdivision of* H if G contains a subdivision of H as an induced subgraph.

A class of graphs $\mathcal{F}$ is said to be χ-*bounded* if there is a function f such that for any graph $G \in \mathcal{F}$, $\chi(G) \leqslant f(\omega(G))$, where $\chi(G)$ and $\omega(G)$ stand for the chromatic number and the clique number of G, respectively.

Scott [7] conjectured that for any graph H, the class of graphs without induced subdivisions of H is χ-bounded, and proved it when H is a tree. But Scott's conjecture was disproved in [6]. Finding which graphs H satisfy the assumption of Scott's conjecture remains a fascinating question. It was proved in [1] that every graph H obtained from the complete graph K_4 by subdividing at least 4 of the 6 edges once (in such a way that the non-subdivided edges, if any, are non-incident), is a counterexample to Scott's conjecture. On the other hand, Scott proved that the class of graphs with no induced subdivision of K_4 has bounded chromatic number (see [5]). Le [4] proved that every graph in this class has chromatic number at most 24. If triangles are also excluded, Chudnovsky et al. [2] proved that the chromatic number is at most 3.

In this paper, we extend the list of graphs known to satisfy Scott's conjecture. Let K_4^+ be the 5-vertex graph obtained from K_4 by subdividing one edge precisely once.

Theorem 1. *The family of graphs with no induced subdivision of K_4^+ is χ-bounded.*

We will need the following result of Kühn and Osthus [3].

Theorem 2 ([3]). *For any graph H and every integer s there is an integer $d = d(H, s)$ such that every graph of average degree at least d contains the complete bipartite graph $K_{s,s}$ as a subgraph, or an induced subdivision of H.*

Proof of Theorem 1. Let k be an integer, let $d(\cdot, \cdot)$ be the function defined in Theorem 2, and let $R(s, t)$ be the Ramsey number of (s, t), i.e. the smallest n such that every graph on n vertices has a stable set of size s or a clique of size t.

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We will prove that every graph G with no induced subdivision of K_4^+, and with clique number at most k, is d-colorable, with $d = \max(k, d(K_4^+, R(4,k)))$. The proof proceeds by induction on the number of vertices of G (the result being trivial if G has at most k vertices). Observe that all induced subgraphs of G have clique number at most k and do not contain any induced subdivision of K_4^+. Therefore, by the induction, we can assume that all induced subgraphs of G are d-colorable. In particular, we can assume that G is connected.

Assume first that G does not contain $K_{s,s}$ as a subgraph, where $s = R(4,k)$. Then by Theorem 2, G has average degree less than d, and hence contains a vertex of degree at most $d-1$. By the induction, $G - v$ has a d-coloring and this coloring can be extended to a d-coloring of G, as desired.

We can thus assume that G contains $K_{s,s}$ as a subgraph. Since G has clique number at most k, it follows from the definition of $R(4,k)$ that G contains $K_{4,4}$ as an induced subgraph. Let M be a set of vertices of G inducing a complete multipartite graph with at least two partite sets containing at least 4 vertices. Assume that among all such sets of vertices of G, M is chosen with maximum cardinality. Let $V_1, V_2, \dots, V_t$ be the partite sets of M.

Let v be a vertex of G, and S be a set of vertices not containing v. The vertex v is *complete* to S if v is adjacent to all the vertices of S, *anticomplete* to S if v is not adjacent to any of the vertices of S, and *mixed* to S otherwise. Let R be the vertices of G not in M. We can assume that R is non-empty, since otherwise G is clearly k-colorable and $k \leqslant d$. We claim that:

If a vertex v of R has at least two neighbors in some set V_i, then it is not mixed to any set V_j with $j \neq i$. (1)

Assume for the sake of contradiction that v has two neighbors a, b in V_i and a neighbor c and a non-neighbor d in V_j, with $j \neq i$. Then v, a, b, c, d induce a copy of K_4^+, a contradiction. This proves (1).

Each vertex v of R has at most one neighbor in each set V_i. (2)

Assume for the sake of contradiction that some vertex $v \in R$ has two neighbors a, b in some set V_i. Then by (1), v is complete or anticomplete to each set V_j with $j \neq i$. Let $\mathcal{A}$ be the family of sets V_j to which v is anticomplete, and let $\mathcal{C}$ be the family of sets V_j to which v is complete. If $\mathcal{A}$ contains at least two elements, i.e. if v is anticomplete to two sets V_j and $V_{j'}$ then by taking $u \in V_j$ and $u' \in V_{j'}$, we observe that v, a, b, u, u' induces a copy of K_4^+, a contradiction. It follows that $\mathcal{A}$ contains at most one element.

Next, we prove that v is complete to V_i. Assume instead that v is mixed to V_i. If v is complete to some set V_ℓ containing at least two vertices, then we obtain a contradiction with (1). It follows that all the elements of $\mathcal{C}$ are singleton. By the definition of M, this implies that $\mathcal{A}$ contains exactly one set V_j, which has size at least 4. Let c be a non-neighbor of v in V_i, and let d, d' be two vertices in V_j. Then v, a, b, c, d, d' is an induced subdivision of K_4^+, a contradiction. We proved that v is complete to V_i. Hence, every set

V_j is either in $\mathcal{A}$ or in $\mathcal{C}$. Since $\mathcal{A}$ contains at most one element, the graph induced by $M \cup \{v\}$ is a complete multipartite graph, with at least two partite sets containing at least 4 elements. This contradicts the maximality of M, and concludes the proof of (2).

$$\text{Each connected component of } G - M \text{ has at most one neighbor in each set } V_i. \tag{3}$$

Assume for the sake of contradiction that some connected component of $G - M$ has at least two neighbors in some set V_i. Then there is a path P whose endpoints u, v are in V_i, and whose internal vertices are in R. Choose P, u, v, V_i such that P contains the least number of edges. Note that by (2), P contains at least 3 edges. Observe also that by the minimality of P, the only edges in G between V_i and the internal vertices of P are the first and last edge of P. Let V_j be a partite set of M with at least 4 elements, with $j \neq i$ (this set exists, by the definition of M). By (2) and the minimality of P, at most two vertices of V_j are adjacent to some internal vertex of P. Since V_j contains at least four vertices, there exist $a, b \in V_j$ that are not adjacent to any internal vertex of P. If V_i has at least three elements then it contains a vertex w distinct from u, v. As w is not adjacent to any vertex of P, the vertices w, a, b together with P induce a subdivision of K_4^+, a contradiction. If V_i has at most two elements, then there must be an integer ℓ distinct from i and j such that V_ℓ has at least four elements. In particular, V_ℓ contains a vertex c that is not adjacent to any internal vertex of P. As a consequence, the vertices a, c together with P induce a subdivision of K_4^+, which is again a contradiction. This proves (3).

Recall that we can assume that R is non-empty. An immediate consequence of (3) is that the neighborhood of each connected component of R is a clique. Since G is connected, it follows that it contains a *clique cutset* K (a clique whose deletion disconnects the graph). Let C be a connected component of $G - K$, let $G_1 = G - C$, and let G_2 be the subgraph of G induced by $C \cup K$. It follows from the induction that there exist d-colorings of G_1 and G_2. Furthermore, since K is a clique, we can assume that the colorings coincide on K. This implies that G is d-colorable and concludes the proof of Theorem 1. $\square$

We remark that we could have used $K_{3,3}$ instead of $K_{4,4}$ in the proof, at the expense of a slightly more detailed analysis. The resulting bound on the chromatic number would have been $\max(k, d(K_4^+, R(3,k)))$ instead of $\max(k, d(K_4^+, R(4,k)))$.

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